The selection of control points for two open non uniform B-spline curves to form Bertrand pairs

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Abstract

In this paper the second and third derivatives of open non- uniform B-spline curves and the Frenet vector fields and curvatures at the points $t = t_d$, $t = t_{m-d}$ and arbitrary point in domain of this curves are given. In addition,the control points of the second open non-uniform B-spline curve are given in terms of the control points of the first open non-uniform B-spline curve when given two curves occured a Bertrand curve pairs at a point.

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1 Introduction

In 1850 J.Bertrand gave the feature that helix curves accept other curves with the same original normal vector field. [1]. The curves that provide this feature are called Bertrand curves.

When the curve with curvature κ and torsion τ in $R^3$is given, if this curve is planar or the relationship between its curvatures $\kappa + a\ \tau = b$ satisfies for nonezero constants a, b then this curve is a Bertrand curve. [2]. It is possible that the Bertrand curves are defined as their principal nornals are parallel. [1]. In recent years, Bertrand curves play an important role in computer-aided geometric designs (CAD) and computer-aided modeling (CAM).[3] , [4], [5] . Due to this importance Bertrand curves have been studied by geometers in different spaces. [6] ,[7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] , [18] ,[19], [20], [21] , [22].

The best examples of points systems are Bezier curves and Bezier surfaces. Bezier and B-Spline curves has been studied in many different are of CAD and CAM system. Some of these studies by G. Farin [23], R. Farouki [24], [25], J. Hoschek [26] , W. Tiller [27] , H. Potmann [28] , Incesu and Gursoy [29] , [30], Samanci et al. [31] , [32], [33], [34] , Baydas and Karakas [35] and Incesu [36] can be given exemplarily.

Other Studies on B-spline curves and NURBS curves [37] , [38], [39], [40], [41], [42], [43], [44], [45], [46] can be given as examples.

NURBS curves are rational B-Spline curves without uniform distribution. Bezier curves, B-Spline curves and NURBS curves are curves that are widely used in computer graphics (CAD) (CAM) systems.

In this study, "When two NUBS curves A and B are given, their control points are b_i and q_i, if these curves form Bertrand pairs at a point, how should be relation between the control points of these curves b_i and q_i?" question has been answered.

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2 Preliminaries

Definition 2.1. The B-spline basis functions of degree d, denoted $N_{i,d}(t)$, defined by the knot vector $t_0, t_1, ..., t_m$ are defined recursively as follows:

$$N_{i,0}(t) = \begin{cases} 1, t \in [t_i, t_{i+1}) \\ 0, \quad otherwise \end{cases}$$

and

$$N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t) \tag{2.1}$$

for $i = 0, ..., n$ and $d \geq 1$. If the knot vector contains a sufficient number of repeated knot values, then a division of the form $N_{i,d-1}(t)/(t_{i+d} - t_i) = 0/0$ (for some i) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$. [47]

The B-spline curve of degree d (or order $d + 1$) with control points $b_0, ..., b_n$ and knots $t_0, ..., t_m$ is defined on the interval $[a, b] = [t_d, t_{m-d}]$ by

$$B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t) \tag{2.2}$$

where $N_{i,d}(t)$ are the B-spline basis functions of degree d. To distinguish Bspline curves from their rational form they are often referred to as integral B-splines.[47]

Theorem 2.2. The B-spline basis functions $N_{i,d}(t)$ satisfy the following properties [47] :

i) **Positivity:** $N_{i,d}(t) > 0$ for $t \in (t_i, t_{i+d+1})$.
ii) **Local Support:** $N_{i,d}(t) = 0$ for $t \notin (t_i, t_{i+d+1})$.
iii) **Piecewise Polynomial:** $N_{i,d}(t)$ are piecewise polynomial functions of degree d.
iv) **Partition of Unity:** $\sum\limits_{i=r-d}^{r} N_{i,d}(t) = 1$ for $t \in [t_r, t_{r+1})$

Theorem 2.3. A B-spline curve defined as (2.2) of degree d defined on the knot vector $t_0, ..., t_m$ satisfies the following properties [47] :

i) **Local Control:** Each segment is determined by $d + 1$ control points. If $t \in [t_r, t_{r+1})(d \leq r \leq m - d - 1)$, then

$$B(t) = \sum_{i=r-d}^{r} b_i N_{i,d}(t).$$

Thus to evaluate $B(t)$ it is sufficient to evaluate $N_{r-d,d}(t), ..., N_{r,d}(t)$.

ii) **Convex Hull:** If $t \in [tr, tr + 1)(d \leq r \leq m - d - 1)$, then $B(t) \in CH\{b_{r-d}, ..., b_r\}$.

iii) **Invariance under Affine Transformations:** Let T be an affine transformation. Then

$$T\left(\sum_{i=r-d}^{r} b_i N_{i,d}(t)\right) = \sum_{i=r-d}^{r} T(b_i)\, N_{i,d}(t)$$

2.1 Open B-spline curves

In general, B-spline curves do not interpolate the first and last control points b_0 and b_n. For curves of degree d, endpoint interpolation and an endpoint tangent condition are obtained by open B-splines. An open B-spline curve is a B-spline curve which exterior knot vectors are the same as the knots t_d and t_{m-d}. i.e. $t_0 = t_1 = ... = t_d$ and $t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ satisfies.

Theorem 2.4. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then

$$B(t_d) = b_0 \quad \text{and} \quad B(t_{m-d}) = b_n$$

satisfies [47].

Definition 2.5. A B-spline curve is said to be **uniform** whenever its knots are equally spaced, and **non-uniform** otherwise. A uniform B-spline curve is said to be **open uniform** whenever its interior knots are equally spaced, and its exterior knots are same. Similarly A non-uniform B-spline curve is said to be **open non-uniform** whenever its exterior knots are same and its interior knots are not equally spaced.

Theorem 2.6. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,

$$B'(t_d) = \frac{d}{t_{d+1} - t_1}(b_1 - b_0) \tag{2.3}$$

$$B'(t_{m-d}) = \frac{d}{t_{m-1} - t_{m-d-1}}(b_n - b_{n-1}) \tag{2.4}$$

are satisfied.[47]

Remark 2.7. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d; t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. If $t_0 = t_1 = ... = t_d = 0$ and $t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m = 1$ Then,

$$B'(0) = \frac{d}{t_{d+1}}(b_1 - b_0) \tag{2.5}$$

$$B'(1) = \frac{d}{1 - t_{m-d-1}}(b_n - b_{n-1}) \tag{2.6}$$

are obtained.

2.2 The De Boor algorithm

Just as the de Casteljau algorithm for B´ezier curve, evaluations of points on a B-spline curve can be performed using a method known as the de Boor algorithm. Let an open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Suppose $t^* \in [t_r, t_{r+1})$.Then, the De Boor algorith can be summarized as follows:

$$\begin{aligned} b_i^j(t) &= \left(1-\alpha_i^j(t)\right) b_{i-1}^{j-1}(t) + \alpha_i^j(t) b_i^{j-1}(t) \\ \alpha_i^j(t) &= \tfrac{t-t_i}{t_{i+d-j+1}-t_i} \end{aligned} \tag{2.7}$$

for $j = 1, ..., d$ and $i = r-d+j, ..., r$.where $b_i^0 = b_i$, $b_{-1} = 0$ and $b_{m-d+1} = 0$.To summarize, for a given parameter value t, the de Boor algorithm (2.7) yields a triangular array of points such that $b_r^d = B(t)$

$$\begin{array}{ccccc} b_{r-d}^0 & b_{r-d+1}^0 & \cdots & \cdots & b_r^0 \\ b_{r-d+1}^1 & \cdots & \cdots & b_r^1 & \\ \cdots & \cdots & & & \\ b_{r-1}^{d-1} & b_r^{d-1} & & & \\ b_r^d = B(t) & & & & \end{array}$$

[47]

3 Main results

3.1 The Frenet frame on the open non-uniform B-spline curves

Theorem 3.1. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,

$$B''(t_d) = \frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2 - b_1) - \frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+1}-t_1)}(b_1 - b_0) \tag{3.1}$$

$$\begin{aligned} B''(t_{m-d}) \quad = \quad & \frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-1}-t_{m-d-1})}(b_n - b_{n-1}) \\ & -\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1} - b_{n-2}) \end{aligned} \tag{3.2}$$

are satisfied.

Proof. From [47], the th r th derivative of an open B-spline curve is $B^{(r)}(t) = \sum_{i=0}^{n-r} b_i^{(r)} N_{i,d-r}^{(r)}(t)$ where $b_i^{(0)} = b_i$ and $b_i^{(r)} = \frac{d-r+1}{t_{i+d+1}-t_{i+r}}(b_{i+1}^{(r-1)} - b_i^{(r-1)})$. According to this $b_1^{(1)} = \frac{d}{t_{d+2}-t_2}(b_2 - b_1)$ and $b_0^{(1)} = \frac{d}{t_{d+1}-t_1}(b_1 - b_0) = B'(t_d)$ can be written. Also from [47],

$B''(t_d) = b_0^{(2)} = \frac{d-1}{t_{d+1}-t_2}(b_1^{(1)} - b_0^{(1)}) = \frac{d-1}{t_{d+1}-t_2}\left[\frac{d}{t_{d+2}-t_2}(b_2 - b_1) - \frac{d}{t_{d+1}-t_1}(b_1 - b_0)\right]$ can be obtained. Similaly the second derivative of open non-uniform B spline curves at the point $t = t_{m-d}$ can be obtained as

$B''(t_{m-d}) = \frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-1}-t_{m-d-1})}(b_n - b_{n-1}) - \frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1} - b_{n-2})$.

Q.E.D.

Theorem 3.2. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,

$$\begin{aligned} B'''(t_d) &= \frac{d(d-1)(d-2)}{(t_{d+1}-t_3)(t_{d+2}-t_3)(t_{d+3}-t_3)}(b_3-b_2) \\ &\quad -\frac{d(d-1)(d-2)(t_{d+1}-t_2+t_{d+2}-t_3)}{(t_{d+1}-t_3)(t_{d+2}-t_2)(t_{d+2}-t_3)(t_{d+1}-t_2)}(b_2-b_1) \\ &\quad +\frac{d(d-1)(d-2)}{(t_{d+1}-t_3)(t_{d+1}-t_2)(t_{d+1}-t_1)}(b_1-b_0) \end{aligned} \tag{3.3}$$

$$\begin{aligned} B'''(t_{m-d}) &= \frac{d(d-1)(d-2)(b_n-b_{n-1})}{(t_{m-3}-t_{m-d-1})(t_{m-2}-t_{m-d-1})(t_{m-1}-t_{m-d-1})} \\ &\quad -\frac{d(d-1)(d-2)(t_{m-3}-t_{m-d-2}+t_{m-2}-t_{m-d-1})(b_{n-1}-b_{n-2})}{(t_{m-3}-t_{m-d-1})(t_{m-2}-t_{m-d-2})(t_{m-2}-t_{m-d-1})(t_{m-3}-t_{m-d-2})} \\ &\quad +\frac{d(d-1)(d-2)(b_{n-2}-b_{n-3})}{(t_{m-3}-t_{m-d-1})(t_{m-3}-t_{m-d-2})(t_{m-3}-t_{m-d-3})} \end{aligned} \tag{3.4}$$

are satisfied.

Proof. Let $r = 3$ be choosen in $b_i^{(r)}$. In this case $b_i^{(3)} = \frac{(d-2)}{(t_{i+d+1}-t_{i+3})}\left(b_{i+1}^{(2)} - b_i^{(2)}\right)$ is obtained. The statements $b_{i+1}^{(2)} = \frac{(d-1)}{(t_{i+d+2}-t_{i+3})}\left(b_{i+2}^{(1)} - b_{i+1}^{(1)}\right)$ and $b_i^{(2)} = \frac{(d-1)}{(t_{i+d+1}-t_{i+2})}\left(b_{i+1}^{(1)} - b_i^{(1)}\right)$ must be substituted in $b_i^{(3)}$. If $b_{i+2}^{(1)} = \frac{d}{(t_{i+d+3}-t_{i+3})}(b_{i+3}-b_{i+2})$, $b_{i+1}^{(1)} = \frac{d}{(t_{i+d+2}-t_{i+2})}(b_{i+2}-b_{i+1})$ and $b_i^{(1)} = \frac{d}{(t_{i+d+1}-t_{i+1})}(b_{i+1}-b_i)$ are substituted in $b_{i+1}^{(2)}$ and $b_i^{(2)}$ then

$b_{i+1}^{(2)} = \frac{(d-1)}{(t_{i+d+2}-t_{i+3})}\left(\frac{d}{(t_{i+d+3}-t_{i+3})}(b_{i+3}-b_{i+2}) - \frac{d}{(t_{i+d+3}-t_{i+3})}(b_{i+3}-b_{i+2})\right)$ and

$b_i^{(2)} = \frac{(d-1)}{(t_{i+d+1}-t_{i+2})}\left(\frac{d}{(t_{i+d+2}-t_{i+2})}(b_{i+2}-b_{i+1}) - \frac{d}{(t_{i+d+1}-t_{i+1})}(b_{i+1}-b_i)\right)$ can be written. So

$$\begin{aligned} b_i^{(3)} &= \frac{d(d-1)(d-2)}{(t_{i+d+1}-t_{i+3})(t_{i+d+2}-t_{i+3})(t_{i+d+3}-t_{i+3})}(b_{i+3}-b_{i+2}) \\ &\quad -\frac{d(d-1)(d-2)(t_{i+d+1}-t_{i+2}+t_{i+d+2}-t_{i+3})}{(t_{i+d+1}-t_{i+3})(t_{i+d+2}-t_{i+2})(t_{i+d+2}-t_{i+3})(t_{i+d+1}-t_{i+2})}(b_{i+2}-b_{i+1}) \\ &\quad +\frac{d(d-1)(d-2)}{(t_{i+d+1}-t_{i+3})(t_{i+d+1}-t_{i+2})(t_{i+d+1}-t_{i+1})}(b_{i+1}-b_i) \end{aligned}$$

is obtained. From end point interpolation property of open B-spline curves $B'''(t_d) = b_0^{(3)}$ and $B'''(t_{m-d}) = b_{n-3}^{(3)}$ satisfy. So the proof is complated. Q.E.D.

Theorem 3.3. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t_d$ are as follows:

$$\begin{aligned} \mathbf{T}(t_d) &= \frac{b_1-b_0}{\|b_1-b_0\|} & \mathbf{B}(t_d) &= \frac{(b_1-b_0)\times(b_2-b_1)}{\|(b_1-b_0)\times(b_2-b_1)\|} \\ \mathbf{N}(t_d) &= \frac{(b_2-b_1)}{\|(b_2-b_1)\|}csc\Phi - \frac{(b_1-b_0)}{\|(b_1-b_0)\|}\cot\Phi & \kappa(t_d) &= \frac{(d-1)(t_{d+1}-t_1)^2\|(b_2-b_1)\|}{d(t_{d+1}-t_2)(t_{d+2}-t_2)\|(b_1-b_0)\|^2}\sin\Phi \end{aligned} \tag{3.5}$$

and

$$\tau(t_d) = \frac{(d-2)\,(t_{d+1}-t_1)\,(t_{d+1}-t_2)\,(t_{d+2}-t_2)\,\|(b_3-b_2)\|\cos\varphi}{d\,(t_{d+1}-t_3)\,(t_{d+2}-t_3)\,(t_{d+3}-t_3)\,\|(b_1-b_0)\|\,\|(b_2-b_1)\|\sin\Phi}$$

whereΦ is the angel between the vectors b_1-b_0 and b_2-b_1 and φ is the angel between the vectors b_3-b_2 and $(b_1-b_0)\times(b_2-b_1)$.

Proof. i) $\mathbf{T}(t_d) = \frac{B'(t_d)}{\|B'(t_d)\|} = \frac{\frac{d}{t_{d+1}-t_1}(b_1-b_0)}{\left\|\frac{d}{t_{d+1}-t_1}(b_1-b_0)\right\|} = \frac{(b_1-b_0)}{\|(b_1-b_0)\|}$

ii) $\mathbf{B}(t_d) = \frac{B'(t_d)\times B''(t_d)}{\|B'(t_d)\times B''(t_d)\|}$
$= \frac{\frac{d}{t_{d+1}-t_1}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2-b_1)}{\left\|\frac{d}{t_{d+1}-t_1}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2-b_1)\right\|} = \frac{(b_1-b_0)\times(b_2-b_1)}{\|(b_1-b_0)\times(b_2-b_1)\|}$

iii) $\mathbf{N}(t_d) = \mathbf{B}(t_d)\times\mathbf{T}(t_d) = \frac{(b_1-b_0)\times(b_2-b_1)}{\|(b_1-b_0)\times(b_2-b_1)\|}\times\frac{(b_1-b_0)}{\|(b_1-b_0)\|}$
$= \frac{((b_1-b_0)\times(b_2-b_1))\times(b_1-b_0)}{\|(b_1-b_0)\times(b_2-b_1)\|\|(b_1-b_0)\|} = \frac{\|(b_1-b_0)\|^2(b_2-b_1)-\langle b_1-b_0,b_2-b_1\rangle(b_1-b_0)}{\|(b_1-b_0)\times(b_2-b_1)\|\|(b_1-b_0)\|}$
$= \frac{(b_2-b_1)}{\|b_2-b_1\|\sin\Phi} - \frac{\cos\Phi(b_1-b_0)}{\sin\Phi\|(b_1-b_0)\|} = \frac{(b_2-b_1)}{\|(b_2-b_1)\|}c\sec\Phi - \frac{(b_1-b_0)}{\|(b_1-b_0)\|}\cot\Phi$

iv) $\kappa(t_d) = \frac{\|B'(t_d)\times B''(t_d)\|}{\|B'(t_d)\|^3} = \frac{\left\|\frac{d}{t_{d+1}-t_1}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2-b_1)\right\|}{\left\|\frac{d}{t_{d+1}-t_1}(b_1-b_0)\right\|^3}$
$= \frac{\frac{d}{t_{d+1}-t_1}\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\|(b_1-b_0)\times(b_2-b_1)\|}{\left(\frac{d}{t_{d+1}-t_1}\right)^3\|(b_1-b_0)\|^3}$
$= \frac{d-1}{d}\frac{(t_{d+1}-t_1)^2}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{\|(b_2-b_1)\|\cos\Phi}{\|(b_1-b_0)\|^2}$

v) Let $\det(b_1-b_0,b_2-b_1,b_3-b_2)$ be denoted K. Then
$\tau(t_d) = \frac{\det\left(B'(t_d),B''(t_d),B'''(t_d)\right)}{\|B'(t_d)\times B''(t_d)\|^2}$
$= \frac{\frac{d}{t_{d+1}-t_1}\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{d(d-1)(d-2)}{(t_{d+1}-t_3)(t_{d+2}-t_3)(t_{d+3}-t_3)}K}{\left\|\frac{d}{t_{d+1}-t_1}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2-b_1)\right\|^2}$
$= \frac{d-2}{d}\cdot\frac{(t_{d+1}-t_1)(t_{d+1}-t_2)(t_{d+2}-t_2)}{(t_{d+1}-t_3)(t_{d+2}-t_3)(t_{d+3}-t_3)}\frac{\|b_3-b_2\|\cos\varphi}{\|(b_1-b_0)\times(b_2-b_1)\|}$ Q.E.D.

Theorem 3.4. An open B-spline curve $B(t)$ of degree d with control points $b_0,b_1,...,b_n$ and knot vectors $t_0=t_1=...=t_d,t_{d+1},...,t_{m-d}=t_{m-d+1}=...=t_{m-1}=t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t=t_{m-d}$ are as follows:

$$\mathbf{T}(t_{m-d}) = \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|} \qquad \mathbf{B}(t_{m-d}) = -\frac{(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|}$$
$$\mathbf{N}(t_{m-d}) = -\frac{(b_{n-1}-b_{n-2})}{\|b_{n-1}-b_{n-2}\|}csc\vartheta + \frac{(b_n-b_{n-1})}{\|b_n-b_{n-1}\|}\cot\vartheta \tag{3.6}$$

and

$$\kappa(t_{m-d}) = \frac{(d-1)\,(t_{m-1}-t_{m-d-1})^2\,\|b_{n-1}-b_{n-2}\|}{d\,(t_{m-2}-t_{m-d-1})\,(t_{m-2}-t_{m-d-2})\,\|b_n-b_{n-1}\|^2}\sin\vartheta$$
$$\tau(t_{m-d}) = -\frac{d-2}{d}\frac{(t_{m-1}-t_{m-d-1})\,(t_{m-2}-t_{m-d-1})\,(t_{m-2}-t_{m-d-2})}{(t_{m-3}-t_{m-d-1})\,(t_{m-3}-t_{m-d-2})\,(t_{m-3}-t_{m-d-3})}\frac{\|(b_{n-2}-b_{n-3})\|\cos\sigma}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|}$$

where ϑ is the angel between the vectors $b_n - b_{n-1}$ and $b_{n-1} - b_{n-2}$ and σ is the angel between the vectors $b_{n-3} - b_{n-2}$ and $(b_n - b_{n-1}) \times (b_{n-1} - b_{n-2})$.

Proof. *i)* $\mathbf{T}(t_{m-d}) = \frac{B'(t_{m-d})}{\|B'(t_{m-d})\|} = \frac{\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})}{\left\|\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\right\|} = \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|}$

ii) $\mathbf{B}(t_{m-d}) = \frac{B'(t_{m-d})\times B''(t_{m-d})}{\|B'(t_{m-d})\times B''(t_{m-d})\|}$

$= -\frac{\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\times\left[\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1}-b_{n-2})\right]}{\left\|\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\times\left[\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1}-b_{n-2})\right]\right\|}$

$= -\frac{(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|}$

iii) $\mathbf{N}(t_{m-d}) = \mathbf{B}(t_{m-d}) \times \mathbf{T}(t_{m-d}) = -\frac{(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|} \times \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|}$

$= \frac{(b_n-b_{n-1})\cos\vartheta}{\|b_n-b_{n-1}\|\sin\vartheta} - \frac{(b_{n-1}-b_{n-2})}{\|(b_{n-1}-b_{n-2})\|\sin\vartheta}$

$= \frac{(b_n-b_{n-1})}{\|b_n-b_{n-1}\|}\cot\vartheta - \frac{(b_{n-1}-b_{n-2})}{\|(b_{n-1}-b_{n-2})\|} c\sec\vartheta$

iv) $\kappa(t_{m-d}) = \frac{\|B'(t_{m-d})\times B''(t_{m-d})\|}{\|B'(t_{m-d})\|^3}$

$= \frac{\left\|\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\times\left[\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1}-b_{n-2})\right]\right\|}{\left\|\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\right\|^3}$

$= \frac{(d-1)(t_{m-1}-t_{m-d-1})^2\|b_{n-1}-b_{n-2}\|}{d(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})\|b_n-b_{n-1}\|^2}\sin\vartheta$

v) Let $\det(b_n - b_{n-1}, b_{n-1} - b_{n-2}, b_{n-2} - b_{n-3})$ be denoted by J. Then

$\tau(t_{m-d}) = \frac{\det\left(B'(t_{m-d}),B''(t_{m-d}),B'''(t_{m-d})\right)}{\|B'(t_{m-d})\times B''(t_{m-d})\|^2}$

$= -\frac{\frac{d}{t_{m-1}-t_{m-d-1}}\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}\frac{d(d-1)(d-2)}{(t_{m-3}-t_{m-d-1})(t_{m-3}-t_{m-d-2})(t_{m-3}-t_{m-d-3})}J}{\left\|\frac{d}{t_{m-1}-t_{m-d-1}}(b_n-b_{n-1})\times\left[\frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1}-b_{n-2})\right]\right\|^2}$

$= -\frac{d-2}{d}\frac{(t_{m-1}-t_{m-d-1})(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}{(t_{m-3}-t_{m-d-1})(t_{m-3}-t_{m-d-2})(t_{m-3}-t_{m-d-3})}\frac{\|(b_{n-2}-b_{n-3})\|\cos\sigma}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|}$ Q.E.D.

In open B-spline curves, in order to express the Frenet frame of the curve $\{T, N, B\}$ and the curvatures at any point $t^* \in (t_r, t_{r+1}), (d \le r \le m-d-1)$, except $t^* = t_d$ and $t^* = t_{m-d}$, the subdivision algorithm is applied to the curve by applying Boor algorithm in parallel with the Casteljau algorithm.Thus the B-spline curve is divided into two segments. The points$\left\{b_r^d, b_r^{d-1}, b_r^{d-2}, b_r^{d-3}\right\}$ found by the algorithm at the given point t^* will represent the first 4 control points of the new B-spline curve on the right of the obtained two segments. So these control points represent the b_0, b_1, b_2, b_3 points of the new B-spline curve. The point t^* here will also represent the point t_d of the new B-spline curve. So following theorem can be proved similarly as before.

Theorem 3.5. An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t^* \in (t_r, t_{r+1})$,$(d \le r \le m-d-1)$ are as follows:

$$\mathbf{T}(t^*) = \frac{b_r^{d-1}-b_r^d}{\left\|b_r^{d-1}-b_r^d\right\|} \qquad \mathbf{B}(t^*) = \frac{\left(b_r^{d-1}-b_r^d\right)\times\left(b_r^{d-2}-b_r^{d-1}\right)}{\left\|\left(b_r^{d-1}-b_r^d\right)\times\left(b_r^{d-2}-b_r^{d-1}\right)\right\|}$$
$$\mathbf{N}(t_d) = \frac{\left(b_r^{d-2}-b_r^{d-1}\right)}{\left\|\left(b_r^{d-2}-b_r^{d-1}\right)\right\|}csc\Phi - \frac{\left(b_r^{d-1}-b_r^d\right)}{\left\|\left(b_r^{d-1}-b_r^d\right)\right\|}\cot\Phi \tag{3.7}$$

and

$$\begin{aligned}
\kappa(t^*) &= \frac{(d-1)\left(t_{d+1}-t_1\right)^2 \left\|\left(b_r^{d-2}-b_r^{d-1}\right)\right\|}{d\left(t_{d+1}-t_2\right)\left(t_{d+2}-t_2\right)\left\|\left(b_r^{d-1}-b_r^{d}\right)\right\|^2}\sin\Phi \\
\tau(t^*) &= \frac{(d-2)\left(t_{d+1}-t_1\right)\left(t_{d+1}-t_2\right)\left(t_{d+2}-t_2\right)\left\|\left(b_r^{d-3}-b_r^{d-2}\right)\right\|\cos\varphi}{d\left(t_{d+1}-t_3\right)\left(t_{d+2}-t_3\right)\left(t_{d+3}-t_3\right)\left\|\left(b_r^{d-1}-b_r^{d}\right)\right\|\left\|\left(b_r^{d-2}-b_r^{d-1}\right)\right\|\sin\Phi}
\end{aligned}$$

whereΦ *is the angel between the vectors* $b_r^{d-1}-b_r^{d}$ *and* $b_r^{d-2}-b_r^{d-1}$ *and* φ *is the angel between the vectors* $b_r^{d-3}-b_r^{d-2}$ *and* $\left(b_r^{d-1}-b_r^{d}\right)\times\left(b_r^{d-2}-b_r^{d-1}\right)$.

3.2 The Bertrand pairs of open non-uniform B-spline curves

Theorem 3.6. Let two open non-uniform B-spline curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and $c_0, c_1, ..., c_n$ respectively and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t = t_d$ if and only if there exist $\theta \in [0, 2\pi]$ and $k \in R$ such that

$$\begin{aligned}
c_1 &= c_0 + (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_1)\sin\theta \\
c_2 &= c_1 + k(b_1 - b_0) + (b_2 - b_1)
\end{aligned}$$

satisfies.

Proof. If these curves γ_1 and γ_2 form a Bertrand pair at the point $t = t_d$ then $\mathbf{N}_{\gamma_1}(t_d) = \mathbf{N}_{\gamma_2}(t_d)$ satisfies. Thus these vectors $((b_1 - b_0) \times (b_2 - b_1)) \times (b_1 - b_0)$ and $((c_1 - c_0) \times (c_2 - c_1)) \times (c_1 - c_0)$ be parallel. So The vectors $(c_1 - c_0) \times (c_2 - c_1)$, $c_1 - c_0$, $(b_1 - b_0) \times (b_2 - b_1)$, and $b_1 - b_0$ must be coplanar. In addition since the vectors system $\{c_1 - c_0 \text{ and } (c_1 - c_0) \times (c_2 - c_1)\}$ and $\{b_1 - b_0 \text{ and } (b_1 - b_0) \times (b_2 - b_1)\}$ are orthogonal, these systems must be $O^+(2)-$equivalent. i.e.

$\{c_1 - c_0 \ , (c_1 - c_0) \times (c_2 - c_1)\} \overset{O^+(2)}{\approx} \{b_1 - b_0 \ , (b_1 - b_0) \times (b_2 - b_1)\}$.

This means that there exist $\theta \in [0, 2\pi]$ such that

$$\begin{aligned}
c_1 - c_0 &= (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_1)\sin\theta \\
(c_1 - c_0) \times (c_2 - c_1) &= (b_1 - b_0)\sin\theta + (b_1 - b_0) \times (b_2 - b_1)\cos\theta
\end{aligned}$$

can be written. From this, $c_1 = c_0 + (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_1)\sin\theta$ is obtained and if this substitude to second then

$(c_1 - c_0) \times (c_2 - c_1) = [(b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_1)\sin\theta] \times (c_2 - c_1)$
$= (b_1 - b_0) \times (c_2 - c_1)\cos\theta - [(b_1 - b_0) \times (b_2 - b_1)] \times (c_2 - c_1)\sin\theta$
$= (b_1 - b_0)\sin\theta + (b_1 - b_0) \times (b_2 - b_1)\cos\theta$

can be written. Thus, from the property of vector product "$\times$" and the linearly independencies of the functions sinus and cosinus,

$$\begin{aligned}
(b_1 - b_0) \times (c_2 - c_1) &= (b_1 - b_0) \times (b_2 - b_1) \\
\langle (c_2 - c_1), (b_2 - b_1) \rangle &= 1 \\
\langle (b_1 - b_0), (c_2 - c_1) \rangle &= 0
\end{aligned}$$

can be obtained. So, the vectors $(c_2-c_1)-(b_2-b_1)$ and (b_1-b_0) must be parallel. Thus, there exist $k\in R$ such that $(c_2-c_1)-(b_2-b_1)=k(b_1-b_0)$ can be written. So

$$c_2=c_1+k(b_1-b_0)+(b_2-b_1)$$

be obtained. Q.E.D.

Theorem 3.7. Let two open non-uniform B-spline curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0,b_1,...,b_n$ and $c_0,c_1,...,c_n$ respectively and knot vectors $t_0=t_1=...=t_d,t_{d+1},...,t_{m-d}=t_{m-d+1}=...=t_{m-1}=t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t=t_{m-d}$ if and only if there exist $\theta\in[0,2\pi]$ and $k\in R$ such that

$$\begin{aligned} c_n &= c_{n-1}+(b_n-b_{n-1})\cos\theta-(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\sin theta \\ c_{n-1} &= c_{n-2}+(b_{n-1}-b_{n-2})+k(b_n-b_{n-1}) \end{aligned}$$

satisfies.

Proof. It is proved similarly as previous theorem. Q.E.D.

Theorem 3.8. Let two open non-uniform B-spline curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0,b_1,...,b_n$ and $c_0,c_1,...,c_n$ respectively and knot vectors $t_0=t_1=...=t_d,t_{d+1},...,t_{m-d}=t_{m-d+1}=...=t_{m-1}=t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t=t^*\in(t_r,t_{r+1})$,$(d\le r\le m-d-1)$ if and only if there exist $\theta\in[0,2\pi]$ and $k\in R$ such that

$$\begin{aligned} c_r^{d-1} &= c_r^d+(b_r^{d-1}-b_r^d)\cos theta-(b_r^{d-1}-b_r^d)\times(b_r^{d-2}-b_r^{d-1}))\sin theta \\ c_r^{d-2} &= c_r^{d-1}+k(b_r^{d-1}-b_r^d)+(b_r^{d-2}-b_r^{d-1}) \end{aligned}$$

satisfies.

Proof. When the De Boor algorithm apply to these curves $\gamma_1(t)$ and $\gamma_2(t)$ at the pıoint $t^*\in(t_r,t_{r+1})$, the control points $\left\{b_r^d,b_r^{d-1},b_r^{d-2},b_r^{d-3}\right\}$ and $\left\{c_r^d,c_r^{d-1},c_r^{d-2},c_r^{d-3}\right\}$ can be obtained. So if these control points be written in the theorem at the point $t=t_d$, then the proof is completed. Q.E.D.

Theorem 3.9. Let two open non-uniform B-spline curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0,b_1,...,b_n$ and $c_0,c_1,...,c_n$ respectively and knot vectors $t_0=t_1=...=t_d,t_{d+1},...,t_{m-d}=t_{m-d+1}=...=t_{m-1}=t_m$ be given. Then if these curves $\gamma_1(t)$ and $\gamma_2(t)$ are $Tr(3)-$ Equivalent curves then γ_1 and γ_2 form a Bertrand pair.

Proof. $Tr(3)$ is a group of all translation's in R^3. A translation g in $Tr(3)$ is defined by $gx=x+b$; $b\in R^3$. Two points x and y in R^3 are called $Tr(3)-$ equivalent if there exist a transformation g in $Tr(3)-$ such that $y=gx$ satisfies. Let two open non-uniform B-spline curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0,b_1,...,b_n$ and $c_0,c_1,...,c_n$ respectively be given. For $p\in R^3$, let $c_i=b_i+p,\ \ i=0,...,n$ be given. it must be proved that $\gamma_1(t)$ and $\gamma_2(t)$ form a Bertrand pair. Fistly in case $t=t_d$ be considered.

$((c_1-c_0)\times(c_2-c_1))\times(c_1-c_0)=(((b_1+p)-(b_0+p))\times((b_2+p)-(b_1+p)))\times((b_1+p)-(b_0+p))$

$=((b_1-b_0)\times(b_2-b_1))\times(b_1-b_0)$ is obtained. So $\mathbf{N}_{\gamma_1}(t_d)=\mathbf{N}_{\gamma_2}(t_d)$ and these curves form a Bertrand pair.

in case $t=t_{m-d}$ be considered. Then

$((c_n-c_{n-1})\times(c_{n-1}-c_{n-2}))\times(c_n-c_{n-1})$

$=(((b_n+p)-(b_{n-1}+p))\times((b_{n-1}+p)-(b_{n-2}+p)))\times((b_n+p)-(b_{n-1}+p))$

$=((b_n-b_{n-1})\times(b_{n-1}-b_{n-2}))\times(b_n-b_{n-1})$ is obtained. So $\mathbf{N}_{\gamma_1}(t_d)=\mathbf{N}_{\gamma_2}(t_d)$ and these curves form a Bertrand pair.

Now let $t^*\in(t_r,t_{r+1})$ be considered $(d\le r\le m-d-1)$. Let apply the De Boor algorihm to these curves at point t^*. it must be proved that $c_i^j=b_i^j+p$ satisfies for every i and every j. Let $\alpha_i^j(t)=\frac{t-t_i}{t_{i+d-j+1}-t_i}$ be considered. Let's do the proof by induction. in case $j=1$. Then

$c_i^1=\left(1-\alpha_i^1(t)\right)c_{i-1}^0(t)+\alpha_i^1(t)c_i^0(t)$

$=\left(1-\alpha_i^1(t)\right)c_{i-1}+\alpha_i^1(t)c_i=c_{i-1}-\alpha_i^1(t)c_{i-1}+\alpha_i^1(t)c_i$

$=(b_{i-1}+p)-\alpha_i^1(t)\,(b_{i-1}+p)+\alpha_i^1(t)\,(b_i+p)$

$=b_{i-1}+p-\alpha_i^1(t)b_{i-1}-\alpha_i^1(t)p+\alpha_i^1(t)b_i+\alpha_i^1(t)p$

$=\left[b_{i-1}-\alpha_i^1(t)b_{i-1}+\alpha_i^1(t)b_i\right]+p\left[\alpha_i^1(t)+1-\alpha_i^1(t)\right]$

$=\left[\left(1-\alpha_i^1(t)\right)b_{i-1}+\alpha_i^1(t)b_i\right]+p$

$=b_i^1+p$

is obtained. Let it is true for $j-1$ be supposed. i.e. $c_i^{j-1}(t)=b_i^{j-1}(t)+p$ is true for ecery i be supposed. Let this be proved in case j.

$$c_i^j(t)=\left(1-\alpha_i^j(t)\right)c_{i-1}^{j-1}(t)+\alpha_i^j(t)c_i^{j-1}(t)$$
$$=\left(1-\alpha_i^j(t)\right)\left(b_{i-1}^{j-1}(t)+p\right)+\alpha_i^j(t)\left(b_i^{j-1}(t)+p\right)$$
$$=b_{i-1}^{j-1}(t)-\alpha_i^j(t)b_{i-1}^{j-1}(t)+p-\alpha_i^j(t)p+\alpha_i^j(t)b_i^{j-1}(t)+\alpha_i^j(t)p$$
$$=\left[b_{i-1}^{j-1}(t)\left(1-\alpha_i^j(t)\right)+\alpha_i^j(t)b_i^{j-1}(t)\right]+p$$
$$=b_i^j(t)+p$$

is obtained. So for every i and for every j , $c_i^j(t)=b_i^j(t)+p$ satisfies. Then $c_r^d=c_r^d+p$, $c_r^{d-1}=b_r^{d-1}$, $c_r^{d-2}=b_r^{d-2}+p$, $c_r^{d-3}=b_r^{d-3}+p$ are written. So it is proved. Q.E.D.

Example 3.10. Let consider the open B-spline curve of degree 3 with control points $b_0=(4,2,2)$, $b_1=(2,1,4)$, $b_2=(3,4,1)$, $b_3=(3,5,5)$ and knot vectors $t_0=t_1=t_2=t_3=0;\;;1=t_4=t_5=t_6=t_7$.

This is cubic B-spline curve. The spline basis functions:

degree 0

$$N_{0,0}=0\quad N_{2,0}=0\quad N_{4,0}=0\quad N_{5,0}=0$$
$$N_{1,0}=0\quad N_{3,0}=\{\}1,\quad t\in[0,1]0,\text{otherwise}\qquad N_{6,0}=0$$

degree 1:

$$N_{0,1}=0\quad N_{2,1}=\{\}1-t,\quad t\in[0,1]0,\text{otherwise}\qquad N_{4,1}=0$$
$$N_{1,1}=0\quad N_{3,1}=\{\}t,\quad t\in[0,1]0,\text{otherwise}\qquad N_{5,1}=0$$

degree 2:

$$N_{0,2}=0 \quad N_{2,2}=\{\}2t\left(1-t\right), \quad t\in[0,1]0, \text{otherwise} \qquad N_{4,2}=0$$
$$N_{1,2}=\{\}\left(1-t\right)^2, \quad t\in[0,1]0, \text{otherwise} \qquad N_{3,2}=\{\}t^2, \quad t\in[0,1]0, \text{otherwise}$$

and degree 3:

$$N_{0,3}=\{\}\left(1-t\right)^3, \quad t\in[0,1]0, \text{otherwise} \qquad N_{2,3}=\{\}3t^2\left(1-t\right), \quad t\in[0,1]0, \text{otherwise}$$
$$N_{1,3}=\{\}3t\left(1-t\right)^2, \quad t\in[0,1]0, \text{otherwise} \qquad N_{3,3}=\{\}t^3, \quad t\in[0,1]0, \text{otherwise}$$

Then the open B-spline curve can be written as:

$$\begin{aligned}\gamma_1(t) &= N_{0,3}b_0+N_{1,3}b_1+N_{2,3}b_2+N_{3,3}b_3 \\ &= \{\}\left(1-t\right)^3 b_0+3t\left(1-t\right)^2 b_1+3t^2\left(1-t\right)b_2+t^3 b_3, \quad t\in[0,1]0, \text{otherwise}\end{aligned}$$

This means: for $t\in[0,1]$,

$$\gamma_1(t)=(-4t^3+9t^2-6t+4,-6t^3+12t^2-3t+2,12t^3-15t^2+6t+2$$

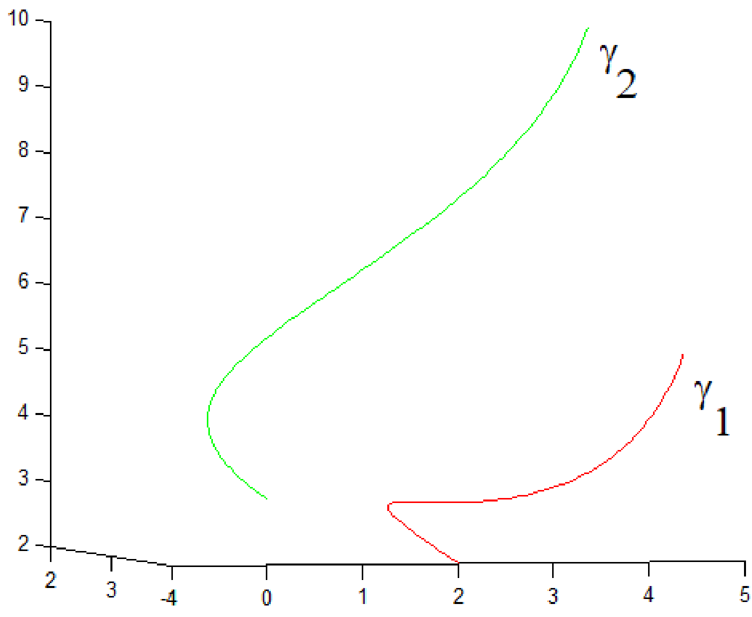

Figure 1. Bertrand pair of open non-uniform B-spline curves γ_1 and γ_2

Now, taking the angels as zero and the multiplicity as 1, from Theorem 10 and 11, the control points of second curve named γ_2 are $c_0=(4,0,3)$, $c_1=(2,-1,5)$, $c_2=(3,3,6)$, $c_3=(3,4,10)$ and these curves form a Bertrand pair indeed.(See Fig. 1)

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